**Conditional probability and independence**

Sometimes you have some prior partial knowledge about the outcome of an experiment. For example, suppose that a friend has flipped two fair coins and has told you that at least one of the coins showed a head. What is the probability that both coins are heads? The information given eliminates the possibility of two tails. The three remaining outcomes are equally likely, and so you infer that each occurs with probability 1/3. Since only one of these outcomes shows two heads, the answer is 1/3.

Conditional probability formalizes the notion of having prior partial knowledge of the outcome of an experiment. The conditional probability of an event A given that another event B occurs is defined to be

Pr{A I B} = (C.16)

whenever Pr O. (Read "Pr r” as “the probability of A given B.") The idea behind equation (C.16) is that since we are given that event B occurs, the event that A also occurs is A. That is Ais the set of outcomes in which both A and B occur. Because the outcome is one of the elementary events in B, we normalize the probabilities of all the elementary events in B by dividing them by Pr { B}, so that they sum to 1. The conditional probability of A given B is, therefore, the ratio of the probability of event Ato the probability of event B. In the example above, A is the event that both coins are heads, and B is the event that at least one coin is a head. Thus, Pr{A I B} = (1/4)/(3/4) = 1/3.

**Bayes's theorem**

From the definition (C.16) of conditional probability and the commutative law A= B, it follows that for two events A and B, each with nonzero probability,

Pr = Pr Pr{A I B} (C.17)

= Pr Pr{B I A}.

Solving for Pr{A I B}, we obtain

Pr{A I B} = , (C.18)

which is known as Bayes's theorem. The denominator Pr{8} is a normalizing constant, which we can reformulate as follows. Since B = (B) U (B), and since Band B are mutually exclusive events,

Pr =+ Pr

= Pr Pr{B I A} + Pr Pr{B I } .

Substituting into equation (C.18) produces an equivalent form of Bayes's theorem:

Pr{A I B} = (C.19)

Bayes's theorem can simplify the computing of conditional probabilities. For example, suppose that you have a fair coin and a biased coin that always comes up heads. Run an experiment consisting of three independent events: choose one of the two coins at random, flip that coin once, and then flip it again. Suppose that the coin you have chosen comes up heads both times. What is the probability that it's the biased coin?

Bayes's theorem solves this problem. Let A be the event that you choose the biased coin, and let B be the event that the chosen coin comes up heads both times. We wish to determine Pr{A I B} knowing that Pr = ½, Pr {B I A} = 1, Pr= 1/2, and Pr{B I } = 1/4. Thus we have

Pr {B I A} =

= 4/5